

Discrete Time Fourier Transform

Maryam Mahsal Khan (Lecturer)
 »B.Sc (CSE) – UET Peshawar
 »M.Sc(EE) – UTP Malaysia

Representation of Sequences by Fourier Transforms

- Many sequences can be represented by a Fourier integral of the form as

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- Frequency response of a LTI system is simply the Fourier transform of the impulse response.

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \quad H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

Representation of Sequences by Fourier Transforms

- The frequency response of discrete-time LTI system is always a periodic function with period 2π .

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = H(e^{j\omega})$$

More generally, $H(e^{j(\omega+2\pi r)}) = H(e^{j\omega})$, for r an integer.

Fourier Transform (Convergence)

- Determining the class of signals that can be represented Fourier transform is equivalent to considering the convergence of the infinite sum of the Fourier transform.
- A **sufficient** condition for convergence can be found as

$$\left| X(e^{j\omega}) \right| = \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]| e^{-j\omega n} \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- Thus, if a sequence is absolutely summable, then its Fourier transform exists. The series can be shown to converge uniformly to a continuous function of ω .
- Since a *stable sequence* is, by definition, *absolutely summable*, all stable sequences have Fourier transforms.

Fourier Transform (Interpretation)

- Signals:** The Fourier Transform $X(e^{j\omega})$ of a signal $x[n]$ describes the **frequency content of the signal**.
 - At each frequency ω_0 , the magnitude spectrum $|X(e^{j\omega_0})|$ describes the amount of that frequency contained in the signal.
 - At each frequency ω_0 , the phase spectrum $\angle X(e^{j\omega_0})$ describes the location (relative shift) of that frequency component of the signal.
- Systems:** The frequency response $H(e^{j\omega})$ of a linear system describes **how frequencies input to the system are modified**:
 - An input frequency component ω_0 is amplified or attenuated by a factor $|H(e^{j\omega_0})|$.
 - An input frequency component ω_0 is shifted by an amount $\angle H(e^{j\omega_0})$.

Example 2.19 Ideal Frequency-Selective Filters: Ideal Lowpass filter

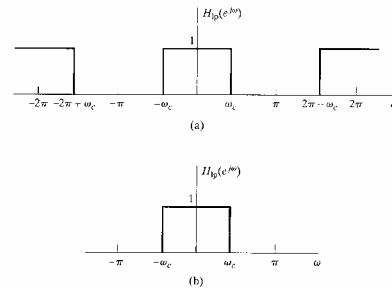


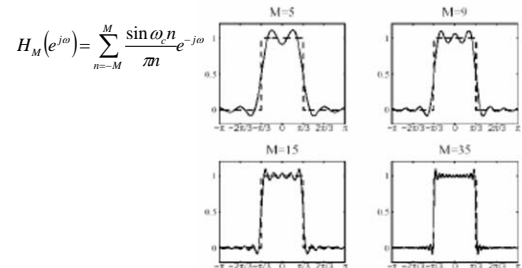
Figure 2.17 Ideal lowpass filter showing (a) periodicity of the frequency response and (b) one period of the periodic frequency response.

DTFT of Ideal Low-Pass Filter

The derivation shows the DTFT of the ideal lowpass filter. The magnitude response is $H_p(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$. The impulse response is $h_p[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_p(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{j2\pi n} [e^{j\omega n}]_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{j2\pi n} = \frac{\sin \omega_c n}{\pi n}$ for $-\infty < n < \infty$. A plot of $h_p[n]$ shows a sinc function centered at $n=0$.

Contd..

- Convergence of the Fourier Transform
- The oscillatory behaviour – Gibbs Phenomena



Frequency-Domain Representation of Discrete-Time Systems

Eigenfunctions for LTI system

- Consider an input sequence

$$x[n] = e^{j\omega n} \text{ for } -\infty < n < \infty,$$

The corresponding output of a LTI discrete-time system with impulse response $h[n]$ is

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} = e^{j\omega n} \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right).$$

- If we define $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$,
- Then we have $y[n] = H(e^{j\omega}) e^{j\omega n}$.
- We define $e^{j\omega n}$ as an eigenfunction of the system, and the associated eigenvalue is $H(e^{j\omega})$.

Frequency-Domain Representation of Discrete-Time Systems (Cont'd)

- Frequency response of the system is defined as

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k},$$

- Real and imaginary representation

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

- Magnitude and phase polar representation

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}.$$

Example 2.17 Frequency Response of the Ideal Delay

- Consider the ideal delay system defined by

$$y[n] = x[n - n_d], \text{ where } n_d \text{ is a fixed integer.}$$

- If we consider $x[n] = e^{j\omega n}$ as input to this system.
- Then we have the output

$$y[n] = e^{j\omega(n-n_d)} = e^{-j\omega n_d} e^{j\omega n}$$

- Therefore, the frequency response of the ideal delay is

$$H(e^{j\omega}) = e^{-j\omega n_d}.$$

- Real and imaginary representation
 $H(e^{j\omega}) = \cos(\omega n_d) + j \sin(\omega n_d)$.
- Magnitude and phase representation
 $|H(e^{j\omega})| = 1$ and $\angle H(e^{j\omega}) = -\omega n_d$.

Example 2.20 Frequency Response of the Moving-Average System

- The impulse response of a moving-average system is

$$h[n] = \begin{cases} \frac{1}{M_1 + M_2 + 1} & -M_1 \leq n \leq M_2, \\ 0 & \text{otherwise.} \end{cases}$$

- The frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-j\omega n} \\ &= \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega M_1} - e^{-j\omega(M_2+1)}}{1 - e^{-j\omega}} \\ &= \frac{1}{M_1 + M_2 + 1} \frac{\sin(\omega(M_1 + M_2 + 1)/2)}{\sin(\omega/2)} e^{-j\omega(M_2 - M_1)/2}. \end{aligned}$$

Example 2.20 Frequency Response of the Moving-Average System (Cont'd)

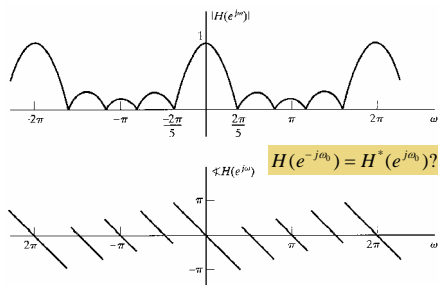


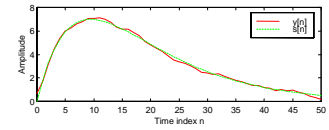
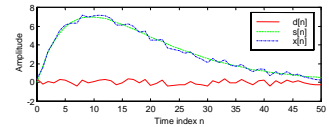
Figure 2.19 (a) Magnitude and (b) phase of the frequency response of the moving-average system for the case $M_1 = 0$ and $M_2 = 4$.

Application Example - Signal Smoothing

A common DSP application is the removal of noise from a signal corrupted by additive noise. A simple 3-point moving average algorithm is given by:

```
% Signal Smoothing by Averaging
clf;
% Generate random noise, d[n]
R = 51; d = 0.8*(rand(R,1) - 0.5);
% Generate uncorrupted signal, s[n]
m = 0:R-1; s = 2*m.*(0.9.^m);
% Generate noise corrupted signal, x[n]
x = s + d';
subplot(2,1,1);
plot(m,d,'r-',m,s,'g--',m,x,'b-');
xlabel('Time index n'); ylabel('Amplitude');
legend('d[n]','s[n]','x[n]');
% do smoothing
x1 = [0 0 x]; x2 = [0 x 0]; x3 = [x 0 0];
y = (x1 + x2 + x3)/3;
subplot(2,1,2);
plot(m,y(2:R+1),'r-',m,s,'g--');
xlabel('Time index n'); ylabel('Amplitude');
```

$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$$



Symmetry Properties of the Fourier Transform

- Symmetry properties of the Fourier transform are often very useful for simplifying the solution of problems.

- Some basic definitions

- Conjugate-symmetric sequence $x_c[n] = x_c^*[-n]$
- Conjugate-antisymmetric sequence $x_o[n] = -x_o^*[-n]$
- Any sequence can be expressed as a sum of a conjugate-symmetric and conjugate-antisymmetric sequence

$$x_c[n] = \frac{1}{2}[x[n] + x^*[-n]] = x_c^*[-n]$$

$$x_o[n] = \frac{1}{2}[x[n] - x^*[-n]] = -x_o^*[-n]$$

$$x[n] = x_c[n] + x_o[n]$$
- Even sequence (real): $x_e[n] = x_e[-n]$
- Odd sequence (real): $x_o[n] = -x_o[-n]$

Symmetry Properties of the Fourier Transform (Cont'd)

- Similarly, a Fourier transform $X(e^{j\omega})$ can be decomposed into a sum of conjugate-symmetric and conjugate-antisymmetric functions.

$$X(e^{j\omega}) = X_c(e^{j\omega}) + X_o(e^{j\omega})$$

$$X_c(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})] = X_c^*(e^{-j\omega})$$

$$X_o(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})] = -X_o^*(e^{-j\omega})$$

$$\text{Re}\{x[n]\} = \frac{1}{2}(x[n] + x^*[n])$$

$$j\text{Im}\{x[n]\} = \frac{1}{2}(x[n] - x^*[n])$$

- For real sequences, the real part of the Fourier transform is an even function, and the imaginary part is an odd function.

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\text{Re}\{x[n]\}$	$X_R(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\text{Im}\{x[n]\}$	$X_A(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \text{Re}\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_A(e^{j\omega}) = j\text{Im}\{X(e^{j\omega})\}$
<i>The following properties apply only when $x[n]$ is real:</i>	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $X(e^{j\omega})$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_A(e^{j\omega}) = -X_A(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$ X(e^{j\omega}) = X(e^{-j\omega}) $ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_A(e^{j\omega})$

Fourier Transform Theorems

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_0]$ (n_0 an integer)	$e^{-jn_0\omega} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} x[n] ^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) ^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

Frequency Response of LCCDE

$$\sum_{k=0}^N a_k Y[n-k] = \sum_{k=0}^M b_k X[n-k]$$

$$\sum_{k=0}^N a_k Y(e^{j\omega})e^{-j\omega k} = \sum_{k=0}^M b_k X(e^{j\omega})e^{-j\omega k}$$

$$\left(\sum_{k=0}^N a_k e^{-j\omega k} \right) Y(e^{j\omega}) = \left(\sum_{k=0}^M b_k e^{-j\omega k} \right) X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\left(\sum_{k=0}^M b_k e^{-j\omega k} \right)}{\left(\sum_{k=0}^N a_k e^{-j\omega k} \right)}$$

Example: Determining an $h[n]$ from Difference Equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$$

$$H(e^{j\omega}) - \frac{1}{2}e^{-j\omega}H(e^{j\omega}) = 1 - \frac{1}{4}e^{-j\omega}$$

$$H(e^{j\omega}) = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} - \frac{\frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

From Transform Tables

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

Example 2.22 Determining the Impulse Response from the Frequency Response

- The frequency response of a high-pass filter with delay

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega n_d} & \omega_c < |\omega| < \pi \\ 0 & |\omega| < \omega_c \end{cases}$$

$$H(e^{j\omega}) = e^{-j\omega n_d} (1 - H_{lp}(e^{j\omega})) = e^{-j\omega n_d} - e^{-j\omega n_d} H_{lp}(e^{j\omega}),$$

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi. \end{cases} \leftrightarrow h_{lp}[n] = \frac{\sin \omega_c n}{\pi n}$$

$$h[n] = \delta[n - n_d] - \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}$$