Sampling of Continuous Time Signals

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Outlines

- Introduction
- Periodic Sampling
- Frequency Domain Representation of Sampling
- Reconstruction from Discrete-Time Samples

Introduction

- **Notations:**

Signals:	**continuous-time**	**discrete-time**
time-domain	$x_c(t)$	$x[n]$
frequency-domain	$X_c(j\Omega)$	$X(e^{j\omega})$
Systems:	**continuous-time**	**discrete-time**
time-domain	$h_c(t)$	$h[n]$
frequency-domain	$H_c(j\Omega)$	$H(e^{j\omega})$

Periodic Sampling

- A typical method of obtaining a discrete-time representation of a continuous-time signal is through periodic sampling.

$$x[n] = x_c(nT), \quad -\infty < n < \infty.$$

  - T is the sampling period
  - $f_s = 1/T$ is the sampling frequency (samples per second)
  - $\Omega_s = 2\pi/T$ is the sampling frequency (radians per second)
- An ideal continuous-to-discrete-time (C/D) converter

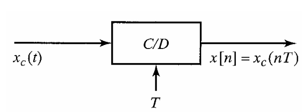
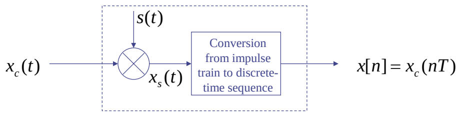


Figure 4.1 Block diagram representation of an ideal continuous-to-discrete-time (C/D) converter.

Mathematical Representation of Sampling



$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \text{(the periodic impulse train)}$$

$$x_s(t) = x_c(t)s(t) = x_c(t)\sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \text{(modulation)}$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) \quad \text{(sifting property)}$$

Periodic Sampling Examples

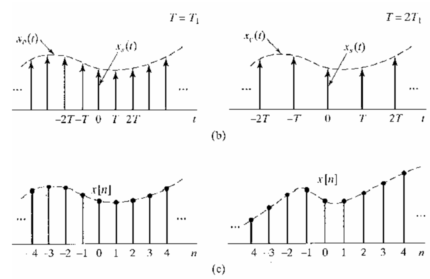


Figure 4.2 Sampling with a periodic impulse train followed by conversion to a discrete-time sequence. (a) Overall system. (b) $x_s(t)$ for two sampling rates. (c) The output sequence for the two different sampling rates.

Frequency-Domain Representation of Sampling: Time-Domain

- We modulate the periodic impulse train with the original continuous-time signals, obtaining

$$\begin{aligned} x_s(t) &= x_c(t)s(t) \\ &= x_c(t)\sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) \end{aligned}$$

Frequency-Domain Representation

- Given the Fourier transform of the impulse train as

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \Leftrightarrow S(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s) \ \text{(where } \Omega_s = \frac{2\pi}{T}\text{)}$$

  - Since

$$x_s(t) = x_c(t)s(t) \Leftrightarrow X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

  - Then

$$\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * \left(\frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)\right) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \end{aligned}$$

Observations of Frequency-Domain Representation of Sampling

- This equation provides the relationship between the Fourier transform of continuous-time signal and discrete-time signal

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

  - $X_s(j\Omega)$ consists of periodically repeated and scaled copies of the Fourier transform of $x_c(t)$, i.e., $X_c(j\Omega)$
  - The copies of $X_c(j\Omega)$ are shifted by integer multiples of the sampling frequency Ω_s.
  - All copies of replicated spectrums are superimposed to produce the Fourier transform of the sampled signal.

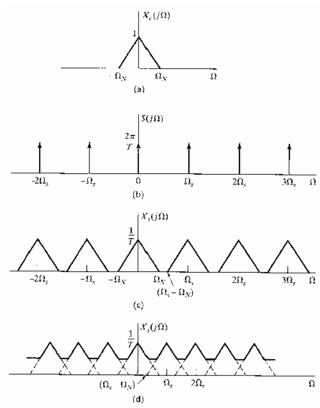


Figure 4.3 Effect in the frequency domain of sampling in the time domain. (a) Spectrum of the original signal. (b) Spectrum of the sampling function. (c) Spectrum of the sampled signal with $\Omega_s > 2\Omega_N$. (d) Spectrum of the sampled signal with $\Omega_s < 2\Omega_N$.

Sampling Rate and Bandwidth

- Given the signal of band-limited

$$|X(j\Omega)| = 0, \quad |\Omega| > \Omega_N$$

- There is no overlap between replicated spectrums, when we have the sampling rate as following

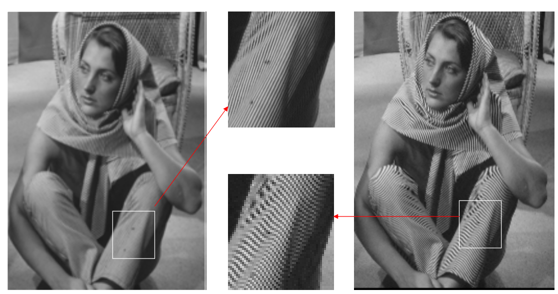
$$\Omega_s > 2\Omega_N$$

  - That means we CAN reconstruct the continuous-time signal with an ideal low-pass filter.
- There will be aliasing distortion, or aliasing when

$$\Omega_s < 2\Omega_N$$

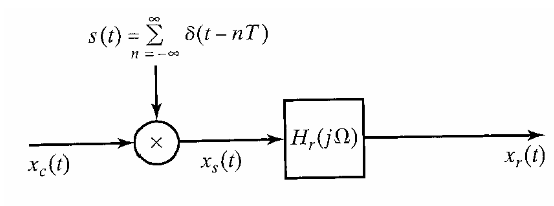
  - That means we CANNOT reconstruct the continuous-time signal from its samples.

What is aliasing?



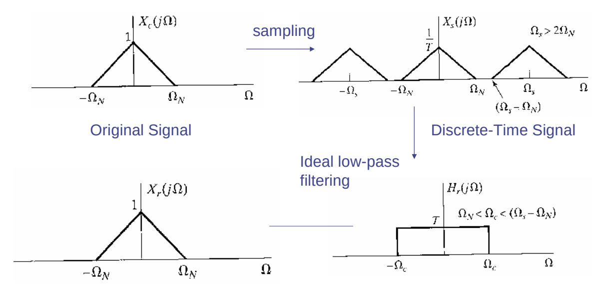
Sampling Without Aliasing   Sampling With Aliasing

How to Reconstruct a Signal?



Ideal low-pass Filtering

How to Reconstruct a Signal? (Cont'd)



Reconstructed Signal

Ideal low-pass Filtering

Sampling and Reconstruction Example

- Given a signal $x_c(t) = \cos\Omega_0 t$
  - What is the Fourier transform of the given signal?
  - Use the **Euler equation**, we know that

$$x_c(t) = \cos\Omega_0 t = \frac{1}{2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right)$$

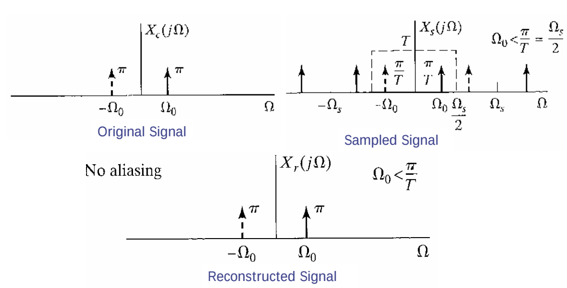
  - According to continuous Fourier transform, we know

$$x(t) = e^{j\Omega_0 t} \Leftrightarrow X(j\Omega) = 2\pi\delta(\Omega - \Omega_0)$$

  - Therefore, the Fourier transform of the given signal is

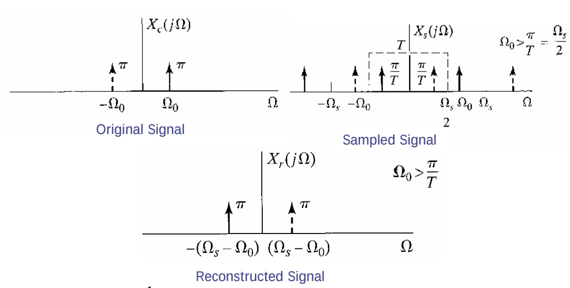
$$X_c(j\Omega) = \pi\left(\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right)$$

Sampling and Reconstruction Example (No Aliasing)



$$x_r(t) = \frac{1}{2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right) = \cos\Omega_0 t$$

Sampling and Reconstruction Example (With Aliasing)



$$x_r(t) = \frac{1}{2}\left(e^{j(\Omega_s - \Omega_0)t} + e^{-j(\Omega_s - \Omega_0)t}\right) = \cos(\Omega_s - \Omega_0)t$$

Nyquist Sampling Theorem

- Suppose that $x_a(t) \leftrightarrow X_a(\Omega)$ is band-limited to a frequency interval $[-\Omega_N, \Omega_N]$, i.e.,

$$X(\Omega) = 0 \text{ for } |\Omega| \geq \Omega_N$$

Then *x*(*t*) can be exactly reconstructed from equidistant samples $x_d[n] = x_a(nT_s) = x_a(2\pi n / \Omega_s)$, if $\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega_N$, where $T_s = 2\pi / \Omega_s$ is the sampling period, Ω_s is the sampling frequency (radians per second), Ω_N is referred to as the *Nyquist frequency*, and $2\Omega_N$ is called the *Nyquist rate*.

How to obtain discrete-time Fourier transform (DTFT)?

- Given the sampled signal as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

- Since we have the following continuous-time Fourier transform (CTFT) pair

$$\delta(t - nT) \Leftrightarrow e^{-j\Omega nT}$$

- Thus we have the continuous-time Fourier transform of the sampled signal as

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn}$$

How to obtain discrete-time Fourier transform (DTFT)? (Cont'd)

- Since we know the relationship between the sampled signal $x_c(nT)$ and the discrete-time sequence $x[n]$

$$x[n] = x_c(nT)$$

- We also have the DTFT of $x[n]$ is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- By comparing with

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn}$$

How to obtain discrete-time Fourier transform (DTFT)? (Cont'd)

- As we compare the following two equations

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega T n}$$

$$X_s(j\Omega) = X(e^{j\omega})\big|_{\omega=\Omega T} = X(e^{j\Omega T}). \qquad X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\big(j(\Omega - k\Omega_s)\big)$$

Continuous-time Fourier transform (CTFT) of the sampled signal $x_s(t)$.

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$(\omega = \Omega T)$

What you can see?

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

Discrete-time Fourier transform (DTFT) of the discrete-time signal $x[n]$.

Example 4.1 (Without Aliasing)

- If we sample the continuous-time signal $x_c(t) = \cos(4000\pi t)$ with sampling period T=1/6000.
  - *Continuous-time Fourier transform* $X_s(j\Omega)$
  - *Discrete-time Fourier transform* $X(e^{j\omega})$
- Problem Analysis
  - Fourier transform of the original signal $\Omega_0 = 4000\pi.$

$$X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi)$$

  - Sampling frequency $\Omega_s = 2\pi / T = 12000\pi.$
  - Fourier transforms of the sampled signal

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \qquad X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{T} - k\frac{2\pi}{T}))$$

Example 4.1 (Cont'd)

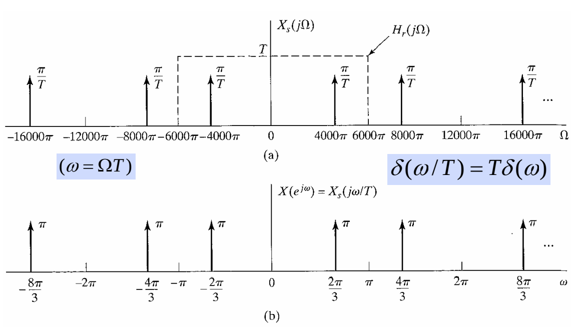


Figure 4.6 Continuous-time (a) and discrete-time (b) Fourier transforms for sampled cosine signal with frequency $\Omega_0 = 4000\pi$ and sampling period $T = 1/6000$.

Example 4.2 (With Aliasing)

- If we sample the continuous-time signal $x_c(t) = \cos(16000\pi t)$ with sampling period T=1/6000.
  - *Continuous-time Fourier transform* $X_s(j\Omega)$
  - *Discrete-time Fourier transform* $X(e^{j\omega})$
- Problem Analysis
  - Fourier transform of the original signal $\Omega_0 = 16000\pi.$

$$X_c(j\Omega) = \pi\delta(\Omega - 16000\pi) + \pi\delta(\Omega + 16000\pi)$$

  - Sampling frequency $\Omega_s = 2\pi / T = 12000\pi.$
  - Fourier transforms of the sampled signal are exactly same as the previous one, why?

$$x[n] = \cos(16000\pi n / 6000) = \cos(2\pi n + 4000\pi n / 6000) = \cos(2\pi n / 3)$$

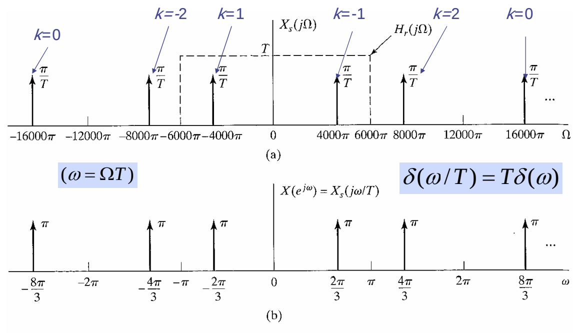


Figure 4.6 Continuous-time (a) and discrete-time (b) Fourier transforms for sampled cosine signal with frequency $\Omega_0 = 16000\pi$ and sampling period $T = 1/6000$.

Example 4.3 (with Aliasing)

- If we sample the continuous-time signal $x_c(t) = \cos(4000\pi t)$ with sampling period T=1/1500.
  - *Continuous-time Fourier transform* $X_s(j\Omega)$
  - *Discrete-time Fourier transform* $X(e^{j\omega})$
- Problem Analysis
  - Fourier transform of the original signal

$$X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi)$$

  - Sampling frequency $\Omega_s = 2\pi / T = 3000\pi.$
  - The discrete-time Fourier transform is the same as previous one. Why?

$$\cos(4000\pi n / 1500) = \cos(2\pi n + 1000\pi n / 1500) = \cos(2\pi n / 3)$$

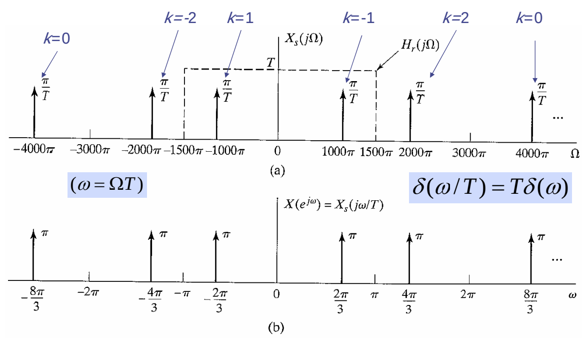


Figure 4.7 Continuous-time (a) and discrete-time (b) Fourier transforms for sampled cosine signal with frequency $\Omega_0 = 4000\pi$ and sampling period $T = 1/1500$.

Reconstruction of a Band-limited Signal from Its Samples

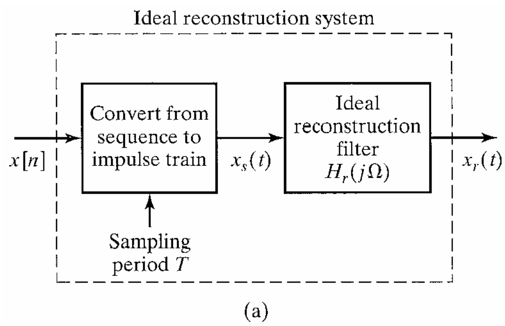
- If the *conditions of the sampling theorem* are met, and if the modulated impulse train is filtered by an appropriate low-pass filter, then the Fourier transform of the filter output will be identical to the Fourier transform of the original signal.
- Given a sequence of samples $x[n]$, we form the impulse train

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

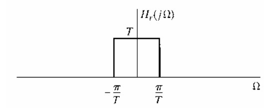
- If the impulse train is the input to an ideal low-pass continuous-time filter with impulse response $h_r(t)$

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_{n=-\infty}^{\infty} x[n]\delta(t - nT) \right) * h_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT)$$

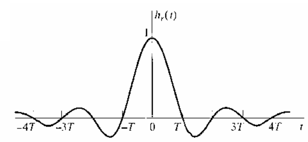
Ideal Reconstruction System



Ideal Reconstruction System (Cont'd)



Frequency Response



Impulse Response

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T} = \text{sinc}\left(\frac{\pi t}{T}\right)$$

Ideal Reconstruction System (Cont'd)

- The ideal reconstruction system is denoted by

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT)$$

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T} = \text{sinc}\left(\frac{\pi t}{T}\right)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

- If $x[n] = x_c(nT)$ and $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T = \Omega_s/2$ then we have

$$x_r(t) = x_c(t)$$

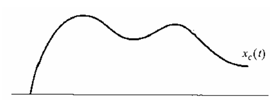
Ideal Band-limited Interpolation

- The ideal low-pass filter interpolates between the impulses of $x[n]$ to construct a continuous-time signal

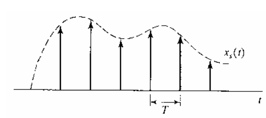
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

- If there is no aliasing, the ideal low-pass filter interpolates correct reconstruction between the samples.
- However, the ideal low-pass filter has infinite length which is not realizable in practice. Finite length low-pass filtering will result in some reconstruction error.

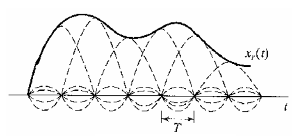
Ideal Band-limited Interpolation (Cont'd)



Original continuous-time signal



Sampled signal



Reconstructed signal

Ideal D/C Converter

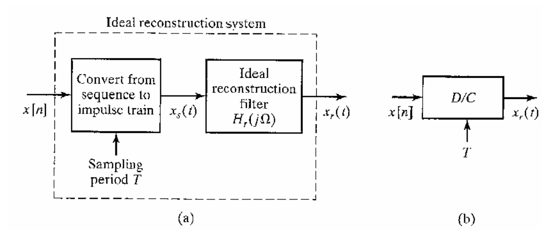


Figure 4.10 (a) Ideal bandlimited signal reconstruction. (b) Equivalent representation as an ideal D/C converter.

Ideal D/C Converter (Cont'd)

- The properties of the ideal D/C converter are most easily seen *in the frequency-domain*.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT) \qquad x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]H_r(j\Omega)e^{-j\Omega Tn}.$$

Linearity of continuous-time Fourier transform

Time shifting leads to an exponential factor in the Fourier transform

$$X_r(j\Omega) = H_r(j\Omega)X(e^{j\Omega T}).$$

Discrete-time Fourier Transform (DTFT) of $x[n]$

Can you get the original signal back?

- The ideal low-pass filter selects the base period of the resulting periodic Fourier transform $X(e^{j\Omega T})$ and compensates for the $1/T$ scaling inherent in sampling.
- If the sequence $x[n]$ has been obtained by sampling a band-limited signal at the Nyquist rate or higher, the reconstructed signal will be equal to the original band-limited signal.
- If there is aliasing during the sampling, the reconstructed signal will be distorted, see Examples 4.2 and 4.3.
- In any case, the output of the ideal D/C converter is always band-limited to at most the cut-off frequency of the low-pass filter, which is taken to one-half the sampling frequency.