Multirate Signal Processing

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Changing the Sampling Rate

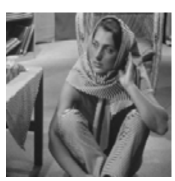
- It is often necessary to change the sampling rate of a discrete time signal, i.e., to obtain a new discrete time representation of the underlying continuous time signals.

$$x[n] = x_c(nT) \quad \text{and} \quad x'[n] = x_c(nT') \quad \text{where } T \neq T'$$

- It is of interest to consider methods of changing the sampling rate that involve discrete time operations.

$$x[n] \Rightarrow x'[n]$$

Sampling Rate Change Examples (Down-sampling)



What happened during down-sampling?

Sampling Rate Reduction by an Integer Factor (Down-sampling)

- The sampling rate of a sequence can be reduced by "sampling it" by defining a new sequence

$$x_d[n] = x[nM] = x_c(nMT).$$

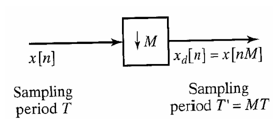


Figure 4.20 Representation of a compressor or discrete-time sampler.

Frequency Representation of Down-Sampling

- First recall that the DTFT of $x[n]=x_c(nT)$ is

$$X(e^{j\omega})=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\left(\frac{\omega}{T}-\frac{2\pi k}{T}\right)\right)$$

- Similarly, the DTFT of $x_d[n]=x[nM]=x_c(nMT)$ is

$$X_d(e^{j\omega})=\frac{1}{MT}\sum_{r=-\infty}^{\infty}X_c\left(j\left(\frac{\omega}{MT}-\frac{2\pi r}{MT}\right)\right)$$

- Questions: what is the relationship between them?

$$X(e^{j\omega})\leftrightarrow X_d(e^{j\omega})$$

Frequency Representation of Down-Sampling (Cont'd)

- We can represent

$$X_d(e^{j\omega})=\frac{1}{MT}\sum_{r=-\infty}^{\infty}X_c\left(j\left(\frac{\omega}{MT}-\frac{2\pi r}{MT}\right)\right)$$

$(r=i+kM)$

$$X_d(e^{j\omega})=\frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\left(\frac{\omega}{MT}-\frac{2\pi(kM+i)}{MT}\right)\right)\right]$$

$$X_d(e^{j\omega})=\frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\left(\frac{\omega}{MT}-\frac{2\pi k}{T}-\frac{2\pi i}{MT}\right)\right)\right]$$

Frequency Representation of Down-Sampling (Cont'd)

- We then have

$$X_d(e^{j\omega})=\frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\left(\frac{\omega-2\pi i}{MT}-\frac{2\pi k}{T}\right)\right)\right]$$

- We know that

$$X(e^{j\omega})=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\left(\frac{\omega}{T}-\frac{2\pi k}{T}\right)\right)\quad\text{(DTFT from CTFT)}$$

$$X\left(e^{j(\omega-2\pi i)/M}\right)=\left[\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\left(\frac{\omega-2\pi i}{MT}-\frac{2\pi k}{T}\right)\right)\right]$$

- Therefore, we have

$$X_d(e^{j\omega})=\frac{1}{M}\sum_{r=0}^{M-1}X\left(e^{j(\omega-2\pi r)/M}\right)$$

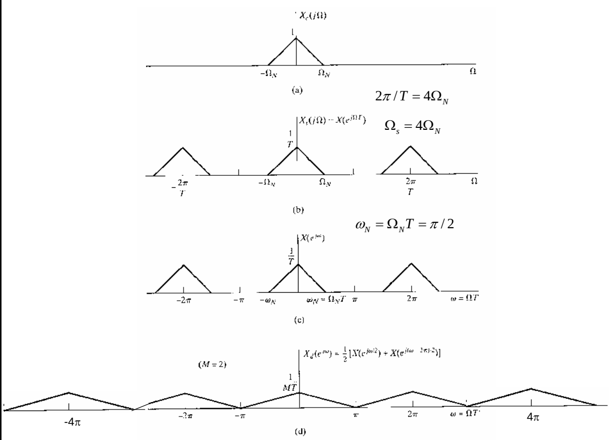
Frequency Representation of Down-Sampling (Cont'd)

- We can have the following conclusions by observing

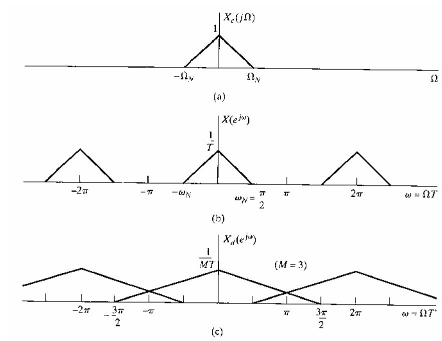
$$X_d(e^{j\omega})=\frac{1}{M}\sum_{r=0}^{M-1}X\left(e^{j(\omega-2\pi r)/M}\right)$$

  - There is a strong analogy between $X_d(e^{j\omega})$ and $X(e^{j\omega})$.
  - $X_d(e^{j\omega})$ can be composed of *M* copies of the periodic Fourier transform $X(e^{j\omega})$, frequency scaled by *M* and shifted by integer multiples of 2π.
  - $X_d(e^{j\omega})$ is periodic with period 2π.
  - Aliasing can be avoided if

$$\begin{cases} X(e^{j\omega})=0,\ \omega_N\le|\omega|\le\pi & \text{(band-limited)} \\ 2\pi/M\ge 2\omega_N & \text{(narrow-banded)} \end{cases}$$



Frequency Representation of Down-Sampling: Example



Down-sampling after Pre-filtering

- If aliasing occurs during down-sampling, we need to reduce the band-width of signal $x[n]$ prior to down-sampling.
- Signal $x[n]$ will be pre-filtered by an ideal low-pass filter with cut-off frequency π/M.

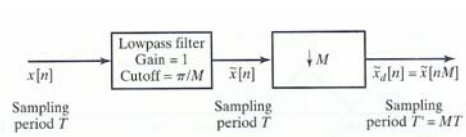
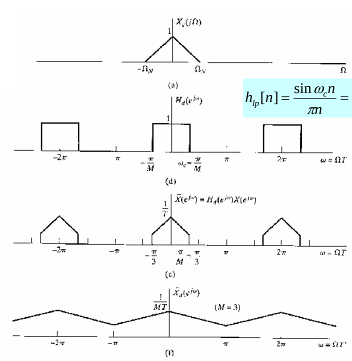


Figure 4.23 General system for sampling rate reduction by M.

Down-sampling after Pre-filtering (Example)



$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n} = \frac{\sin(\pi n / M)}{\pi n}$$

Increasing the Sampling Rate by an Integer Factor

- Consider a signal $x[n]$ whose sampling rate we wish to increase by a factor of L (up-sampling).

$$x_e[n] = \begin{cases} x[n/L], n = kL \text{ and } k \in Z \\ 0, \text{ otherwise} \end{cases}$$

- For example, $x[n] = (1 \quad 2 \quad 3 \quad 4 \quad 5)$

$$(n = 0, 1, 2, 3, 4)$$

$$x_e[n] = (1 \quad 0 \quad 2 \quad 0 \quad 3 \quad 0 \quad 4 \quad 0 \quad 5 \quad 0)$$

$$(n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \text{ and } L = 2)$$

- Or equivalently,

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - kL]$$

Frequency Representation of Up-Sampling

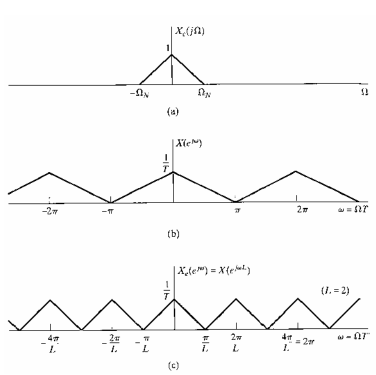
- The Fourier transform (DTFT) of the up-sampled signal is

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}\left(x[k]\delta[n-kL]\right)\right)e^{-j\omega n}$$

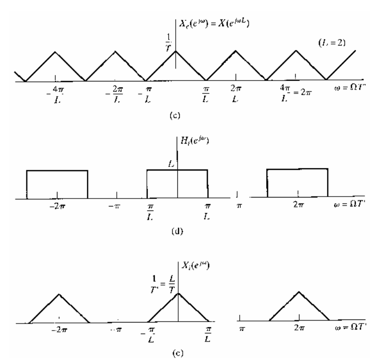
$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega kL} = X(e^{j\omega L})$$

- We can see the Fourier transform of the output of the expander is a frequency-scaled version of the Fourier transform of the input, i.e, ω is replaced by ωL.

Frequency Representation of Up-Sampling (Example)



Frequency Representation of Up-Sampling and Ideal Low-pass Filtering (Example)



General System for Up-sampling

- To fill missing samples, the operation of up-sampling is therefore considered to be synonymous with interpolation.

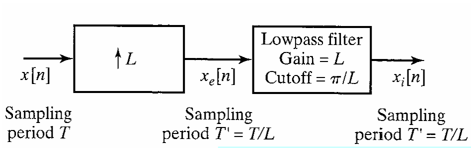


Figure 4.24 General system for sampling rate increase by L.

$$h_{lp}[n] = L \cdot \left(\frac{\sin \omega_c n}{\pi n} \right) = \frac{\sin \pi n / L}{\pi n / L}$$

Gain

Ideal Low-pass Filtering after Up-sampling

- As in the case of D/C converter, it is possible to obtain an interpolation formula with an ideal low-pass filter as

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] h_{lp}[n-kL] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n-kL)/L]}{\pi(n-kL)/L}$$

- The impulse response of the low pass filter has properties

$$h_i[n] = \frac{\sin(\pi n / L)}{\pi n / L} \Rightarrow \begin{array}{l} h_i[0] = 1 \\ h_i[n] = 0, n = \pm L, \pm 2L, \pm 3L, ... \end{array}$$

- Thus for the ideal low-pass interpolation filter, we have

$$x_i[n] = x[n/L], \quad n = \pm L, \pm 2L, \pm 3L, ...$$

Linear Interpolation after Up-sampling

- Linear interpolation can be accomplished by the

$$h_{\text{lin}} = \begin{cases} 1 - |n|/L, & |n| \le L \\ 0, & \text{otherwise} \end{cases}$$

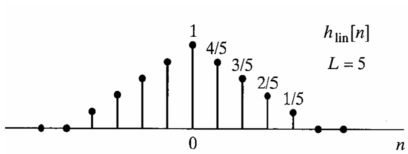
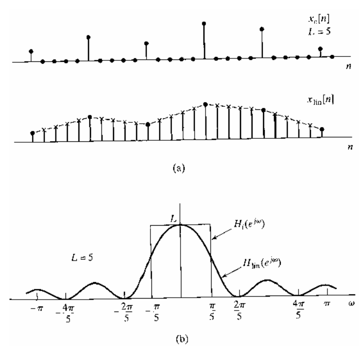


Figure 4.26 Impulse response for linear interpolation.

Linear Interpolation after Up-sampling (Example)



$$H_{\text{lin}}(e^{j\omega}) = \frac{1}{L}\left[\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right]^2$$

Figure 4.27 (a) Illustration of linear interpolation by filtering. (b) Frequency response of linear interpolator compared with ideal lowpass interpolation filter.

Linear Interpolation after Up-sampling (Example, Cont'd)

- Please note that

$$h_{\text{lin}}[n] = \begin{cases} 1-|n|/L, & |n| \le L \\ 0, & \text{otherwise} \end{cases} \Rightarrow \begin{matrix} h_i[0]=1 \\ h_i[n]=0\,, n=\pm L, \pm 2L, \pm 3L, ... \end{matrix}$$

- So that $x_i[n] = \sum_{k=-\infty}^{\infty} x[k] h_{\text{lin}}[n-kL]$

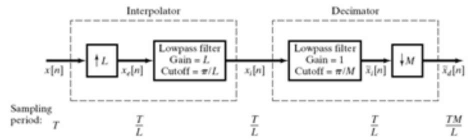
$$x_{\text{lin}}[n] = x[n/L], \quad n = \pm L, \pm 2L, \pm 3L, ...$$

- The amount of distortion in the intervening samples can be gauged by comparing the frequency response of the linear interpolator with that of the ideal low-pass interpolator, as

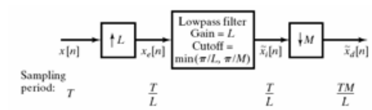
$$H_{\text{lin}}(e^{j\omega}) = \frac{1}{L}\left[\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right]^2$$

Changing the Sampling Rate by Non-Integer Factor

- Combine decimation and interpolation for non-integer factors



- The two low-pass filters can be combined into a single one



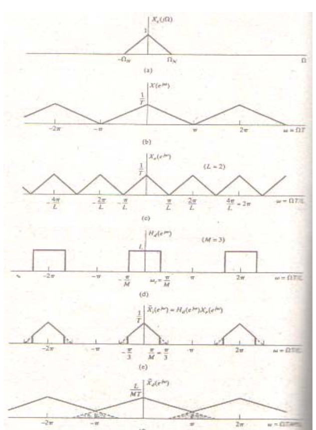


Figure 4.29 Illustration of changing the sampling rate by a noninteger factor.