Filter Design Techniques (FIR Filters)

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Outlines

- FIR Filter Design Basic
- Design of FIR Filters by Windowing
- Design of FIR Filters by Kaiser Windowing

Introduction

- Given a set specifications or stated constraints on
  - Magnitude spectrum
  - Phase spectrum
- Find $\{a_m\}$ and $\{b_k\}$ where,

$$H(z) = \frac{\sum_{k=0}^{M} a_k z^{-k}}{\sum_{k=0}^{N} b_k z^{-k}}$$

FIR Filter Design Basic

- Here it is assumed that $b_1 = b_2 = b_3 = \cdots = b_N = 0$
- Hence

$$H(z) = \sum_{m=0}^{M} a_m z^{-m}$$

  - And so the unit pulse response of the filter is clearly:

$$h[n] = \begin{cases} a_n; & 0 \le n \le M \\ 0; & \text{else} \end{cases}$$

- Problem: Given specifications on $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$,
  - find

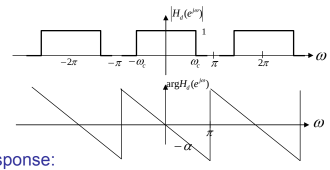
$$\{a_n; n = 1, \ldots, M\}$$

- FIR filters are often called non-recursive for obvious reasons.

Filter Specifications

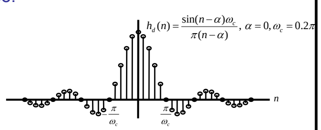
- A set of filter specifications must be defined before a filter can be designed.
- Recall the ideal lowpass filter with linear phase and a cutoff frequency ω_c

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$



- which has a unit sample response:

$$h_d(n) = \frac{\sin(n-\alpha)\omega_c}{\pi(n-\alpha)}$$

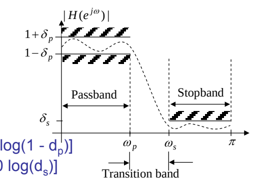


Note: $h_d(n) = \frac{\sin(n-\alpha)\omega_c}{\pi(n-\alpha)} \quad \left[= \frac{\omega_c}{\pi} \mathrm{sinc}\left((n-\alpha)\frac{\omega_c}{\pi} \right) \right]$

Filter Specifications

- Because the ideal filter is unrealizable (noncausal and unstable), we relax the ideal constraints on the frequency response.
- Typical specifications of a lowpass filter have the form:

$$1-\delta_p < |H(e^{j\omega})| \le 1+\delta_p, \quad 0 \le |\omega| < \omega_p$$
$$|H(e^{j\omega})| \le \delta_s, \quad \omega_s \le |\omega| < \pi$$



- Specifications include:
  - 1. Passband cutoff frequency, ω_p
  - 2. Stopband cutoff frequency, ω_s
  - 3. Passband deviation, α_p [or ap dB = -20 log(1 - d_p)]
  - 4. Stopband deviations, α_s [or as dB = -20 log(d_s)]

The interval $[\omega_p, \omega_s]$ is called the transition band.

In Matlab, passband ripple, Rp (dB) = -20 log (1-δ_p)
stopband ripple, Rs (dB) = -20 log δ_s

FIR Filter: Advantages and Disadvantages

- **Advantages:**
  - Always stable (assume non-recursive implementation).
  - Quantization noise is not much of a problem.
  - Can be designed to have exact linear phase even when causal, while meeting a prescribed phase to arbitrary accuracy.
  - Design methods generally linear.
  - Can be realised efficiently in hardware
- **Disadvantages:**
  - A high-order filter is generally needed to satisfy the stated specification – so more coefficients are needed with more storage and computation – leads to longer delays.

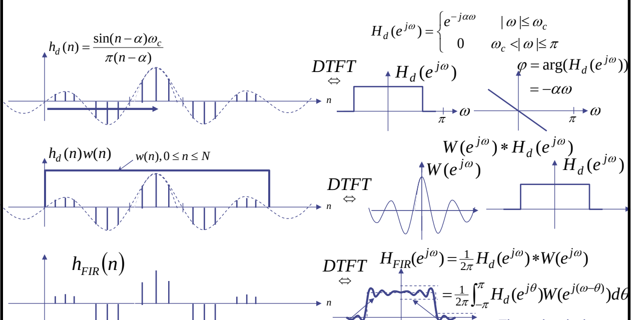
Design FIR Filters using Windows

- *The frequency response of an N th-order causal FIR filter is:*

$$H(e^{j\omega}) = \sum_{n=0}^{N} h(n) e^{-jn\omega}$$

- *Design of an FIR filter involves finding the coefficients h(n) that result in a frequency response that satisfies the given filter specifications.*
- Let $h_d(n)$ be the unit sample response of an ideal frequency selective filter with linear phase, $H_d(e^{j\omega}) = A(e^{j\omega}) e^{-j(\alpha\omega-\beta)}$
- Generally $h_d(n)$ is infinite in length; an FIR approximation to $H_d(e^{j\omega})$ can be found by the window design method.
- Here, the FIR filter is designed by windowing (make it finite in length) and delaying (make it causal) the unit sample response: $h_{FIR}(n) = h_d(n) w(n)$
- where w(n) is a finite-length window that is equal to zero outside the interval $0 \le n \le N$ and is symmetric about the midpoint: $w(n) = w(N-n)$
- Note: N = filter order.

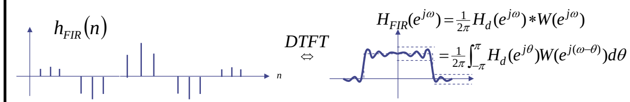
Lowpass FIR Filter Design
- Effect of Window on Frequency Response



The effect of the window (impulse response delayed by α for and truncated by a window w(n) to make impulse response causal) on the frequency response results in smoothing (with ripples) of the ideal frequency response - *Gibbs phenomenon*

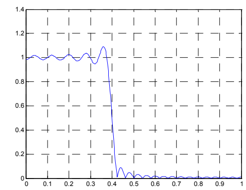
Thus, the design requires the ideal filter specifications to be relaxed.

Linear Phase Lowpass FIR Filter Design
- Effect of Window on Frequency Response

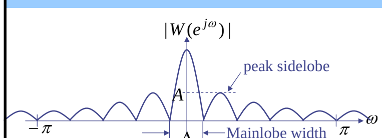


% A length 51 filter with a lowpass cutoff ω_c of 0.4π rad/sec

```
b=0.4*sinc(0.4*(-25:25));
[H,w]=freqz(b,1,512,2);
plot(w,abs(H)),grid
```



Linear Phase FIR Filter Design using Windows
- window considerations



How well the frequency of a filter designed with the window method approximates a desired response, $H_d(e^{j\omega})$, is determined by:

1. Mainlobe width
2. Peak sidelobe amplitude, A

General properties of windows:

1. As the length, N of the window increases, the width of the main lobe decreases, which results in a decrease in the transition width between passbands and stopbands. This relationship is approximated by : $N \propto \frac{1}{\Delta f} \quad \Rightarrow N\Delta f = c$

where Δf is the transition width, and *c* is a parameter that depends on the window.

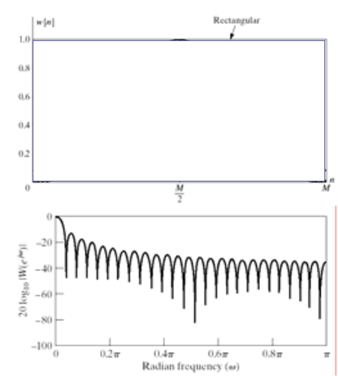
2. The peak sidelobe amplitude of the window is determined by the shape of the window (independent of the window length).

3. If the window shape is changed to decrease the sidelobe amplitude, the width of the mainlobe will generally increase.

Rectangular Window

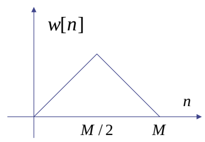
$$w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & else \end{cases}$$

- Narrowest main lobe
  - $4\pi/(M+1)$
  - Sharpest transitions at discontinuities in frequency
- Large side lobes
  - -13 dB
  - Large oscillation around discontinuities

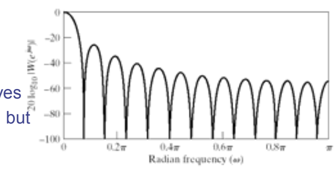


Bartlett (Triangular) Window

$$w[n]=\begin{cases} 2n/M & 0\le n\le M/2 \\ 2-2n/M & M/2\le n\le M \\ 0 & else \end{cases}$$

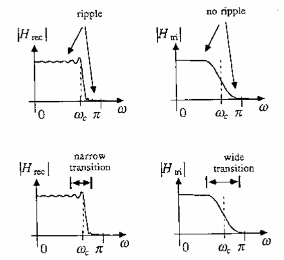


- Medium main lobe
  - $8\pi/M$
- Side lobes
  - -25 dB
- using the triangular window gives an approximation with smooth, but wider transition.



Windowing: Trade-off

- Two major distortions: Ripples vs. Transition Width
  - *Rectangular window* has a sharp transition but severe ripple.
  - *Triangular window* has no ripple but a very wide transition.



Other Windows

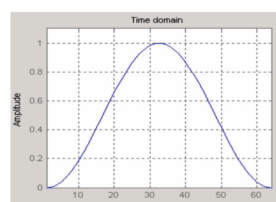
- Other windows attempt to optimize this trade-off. Widely used windows that give intermediate results are:
  - Hamming Window: $w[n]=\begin{cases} 0.54-0.46\cos(2\pi n/M); & 0\le n\le M \\ 0; & else \end{cases}$
  - Hanning Window: $w[n]=\begin{cases} 0.5-0.5\cos(2\pi n/M); & 0\le n\le M \\ 0; & else \end{cases}$
  - Blackman Window:

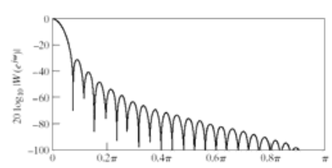
$$w[n]=\begin{cases} 0.42-0.5\cos(2\pi n/M)+0.08\cos(4\pi n/M); & 0\le n\le M \\ 0; & else \end{cases}$$

Hanning Window

$$w[n]=\begin{cases} \frac{1}{2}\left[1-\cos\left(\frac{2\pi n}{M}\right)\right] & 0\le n\le M \\ 0 & else \end{cases}$$

- Medium main lobe
  - $8\pi/M$
- Side lobes
  - -31 dB
- Hamming window performs better
- Same complexity as Hamming

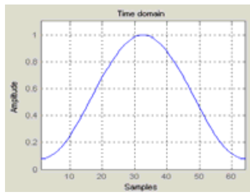


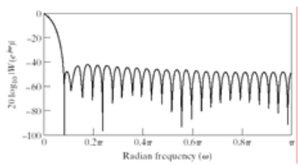


Hamming Window

$$w[n]=\begin{cases}0.54-0.46\cos\left(\frac{2\pi n}{M}\right) & 0\le n\le M\\ 0 & else\end{cases}$$

- Medium main lobe
  - $8\pi/M$
- Good side lobes
  - -41 dB
- Simpler than Blackman

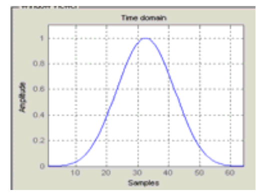


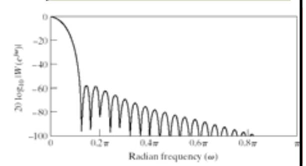


Blackman Window

$$w[n]=\begin{cases}0.42-0.5\cos\left(\frac{2\pi n}{M}\right)+0.08\cos\left(\frac{4\pi n}{M}\right) & 0\le n\le M\\ 0 & else\end{cases}$$

- Large main lobe
  - $12\pi/M$
- Very good side lobes
  - -57 dB
- Complex equation





Windowing Comparisons: Transition Width – Ripples

Window	Peak Sidelobe Amplitude (dB)	Mainlobe Width
	(ripples)	(transition width)
Rectangular	-13	$4\pi/(M+1)$
Triangular	-25	$\approx 8\pi/M$
Hanning	-31	$\approx 8\pi/M$
Hamming	-41	$\approx 8\pi/M$
Blackman	-57	$\approx 12\pi/M$

Rectangular: transition width is optimized.
Blackman: Ripple is minimized.

Windowing Filter Design Examples (Cont'd)

TABLE 7.1 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	$\Delta\omega_m$ Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hanning	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

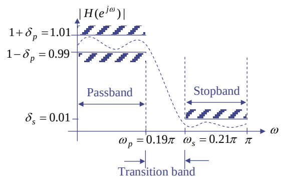
Linear Phase FIR Filter Design using Windows - Design Example

Example 1 — Design an FIR linear phase lowpass based on the following specifications:

$$0.99 < |H(e^{j\omega})| \le 1.01 \quad 0 \le |\omega| < 0.19\pi$$

$$|H(e^{j\omega})| \le 0.01 \qquad 0.21\pi \le |\omega| < \pi$$

Solution 1



For a stopband attenuation of 20 log(0.01) = -40 dB, we may use a Hanning window. Although we could use Hamming or Blackman window, these windows would overdesign the filter and produce a larger stopband attenuation at the expense of an increase in the transition width.

Linear Phase FIR Filter Design using Windows - Design Example

Solution 1 cont.

Since the specification calls for a transition width of $\Delta\omega = \omega_s - \omega_p = 0.02\pi$, or $\Delta f = 0.01$ with $N\Delta f = 3.1$ (from table)

for a Hanning window (see table), an estimate of the required filter order is:

$$N = 3.1/\Delta f = 310$$

The last step is to find the unit sample response of the ideal lowpass filter that is to be windowed.

With a cutoff frequency of $\omega_c = (\omega_s + \omega_p)/2 = 0.2\pi$, and a delay of $\alpha = N/2 = 155$,

the ideal unit sample response is: $h_d(n) = \dfrac{\sin(n-\alpha)\omega_c}{\pi(n-\alpha)} = \dfrac{\sin[0.2\pi(n-155)]}{(n-155)\pi}$

Thus, the FIR filter coefficients (filter unit sample response) is:

$$h_{FIR}(n) = h_d(n) w_{Hanning}(n) = \left\{\frac{\sin[0.2\pi(n-155)]}{(n-155)\pi}\right\}\left\{0.54 - 0.46\cos(\frac{2\pi n}{N})\right\}$$

Incorporation of Generalized Linear Phase

- Windows are designed with linear phase in mind
  - Symmetric around M/2

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & else \end{cases}$$

- So their Fourier transform are of the form

$$W(e^{j\omega}) = W_e(e^{j\omega})e^{-j\omega M/2} \quad \text{where } W_e(e^{j\omega}) \text{is a real and even}$$

- Will keep symmetry properties of the desired impulse response
- Assume symmetric desired response

$$H_d(e^{j\omega}) = H_e(e^{j\omega})e^{-j\omega M/2}$$

- With symmetric window

$$A_e(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_e(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

  - Periodic convolution of real functions

Linear-Phase Lowpass filter

- Desired frequency response

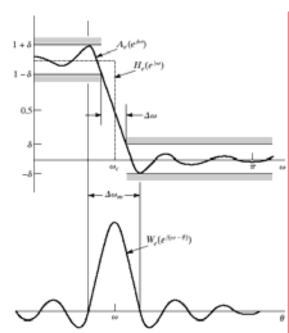
$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

- Corresponding impulse response

$$h_{lp}[n] = \frac{\sin[\omega_c(n-M/2)]}{\pi(n-M/2)}$$

- Desired response is even symmetric, use symmetric window

$$h[n] = \frac{\sin[\omega_c(n-M/2)]}{\pi(n-M/2)}w[n]$$



Windowing Filter Design Examples

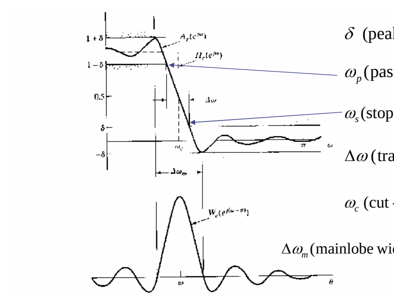


Figure 7.23 Illustration of type of approximation obtained at a discontinuity of the ideal frequency response.

δ (peak approximation error)

ω_p (passband cut-off frequency)

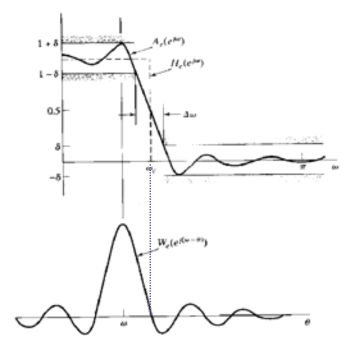
ω_s (stop-band cut-off frequency)

$\Delta\omega$ (transition width) $= \omega_s - \omega_p$

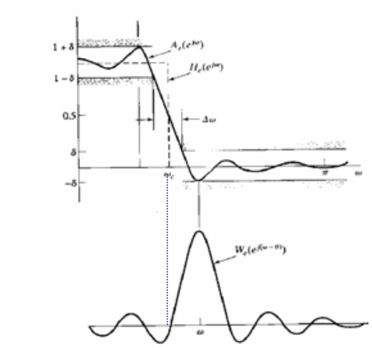
ω_c (cut-off frequency) $= \dfrac{\omega_s + \omega_p}{2}$

$\Delta\omega_m$ (mainlobe width) $> \Delta\omega$ (transition width)

Overshoot in the Pass-band



Undershoot in the Stop-band



Kaiser Window Filter Design Method

- Family of windows developed by Kaiser are defined as follows:

$$w(n) = \frac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)} \qquad 0 \le n \le N$$

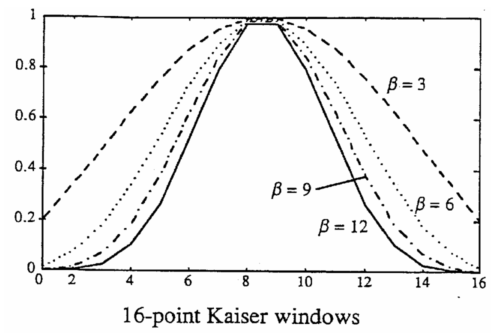
- where $\alpha = N/2$, and $I_0(.)$ is a zeroth-order modified Bessel function of the first kind, which may be easily generated using the power series expansion

$$I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{(x/2)^k}{k!}\right]^2$$

- The parameter β determines the shape of the window i.e. it controls the trade-off between the mainlobe width and the sidelobe amplitude.

- A Kaiser window is nearly optimum in the sense of having the most energy in its mainlobe for a given sidelobe amplitude.

Impulse Response of Kaiser Windowing



Characteristics of the Kaiser Window

Characteristics of the Kaiser window as a function of β

Parameter(β)	Sidelobe(dB)	Transition Width(NΔf)	Stopband Attenuation(dB)
2.0	-19	1.5	-29
3.0	-24	2.0	-37
4.0	-30	2.6	-45
5.0	-37	3.2	-54
6.0	-44	3.8	-63
7.0	-51	4.5	-72
8.0	-59	5.1	-81
9.0	-67	5.7	-90
10.0	-74	6.4	-99

Kaiser Windows - Design parameters

- Two empirically derived relationships for the Kaiser window that facilitate the use of these windows to design FIR filters:

1. The first relates the stopband ripple of a lowpass filter, $\alpha_s = -20\log(\delta_s)$ to the parameter β,

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7) & \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) & 21 \le \alpha_s \le 50 \\ 0.0 & \alpha_s < 21 \end{cases}$$

2. The second relates N to the transition width Δf and the stopband attenuation α_s,

$$N = \frac{\alpha_s - 7.95}{14.36\Delta f} \qquad \alpha_s \ge 21$$

Note that if α_s < 21 dB, a rectangular window may be used (β = 0), and N = 0.9/ Δf.

Kaiser Windows - Design Example

Example 2 Design a lowpass filter with the following requirements:

Cutoff frequency, $\omega_c = \pi/4$

Transition width, $\Delta\omega = 0.02\pi$

Stopband ripple, $\delta_s = 0.01$

Solution Because $\alpha_s = -20\log(0.01) = -40dB$

the Kaiser window parameter is: $\beta = 0.5842(40-21)^{0.4} + 0.07886(40-21) = 3.4$

With $\Delta f = \Delta\omega / 2\pi = 0.01$

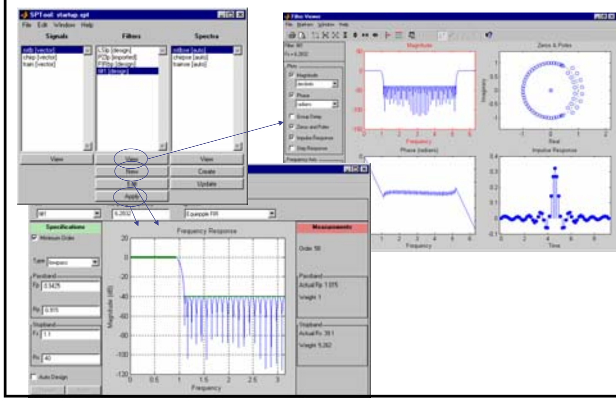
$$N = \frac{40 - 7.95}{14.36(0.01)} = 224$$

Therefore

$$h_{FIR}(n) = h_d(n)w(n) = \frac{\sin(n-\alpha)\omega_c}{\pi(n-\alpha)} \cdot \frac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}$$

$$= \left\{\frac{\sin[(n-112)\pi/4]}{(n-112)\pi}\right\}\left\{\frac{I_0[3.4(1-[(n-112)/112]^2)^{1/2}]}{I_0(3.4)}\right\}$$

Kaiser Windows Filter Design Method – MATLAB Design Example using sptool



Kaiser Windows Filter Design Method – MATLAB Design Example using sptool

Example 3 Design a lowpass filter with passband from 0 to 1 KHz and stopband from 1500 Hz to 4 KHz. Specify passband ripple of 5% and stopband attenuation of 40 dB.

Solution 3 Since stopband ends at 4 kHz, it is implied that sampling freq. is 8 kHz. Use *sptool* and input the above. parameters.

$f_{samp} = 8kHz$

$F_p = 1kHz, \quad F_s = 1.5kHz$

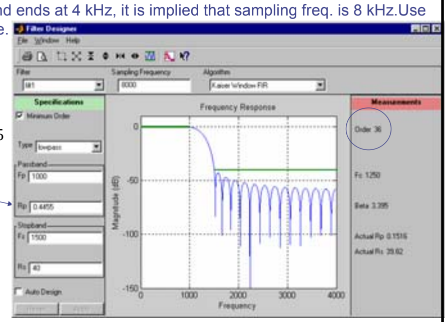
$\text{Passband ripple} = 5\% = 0.05$

$\Rightarrow R_p = -20\log(1-0.05)$

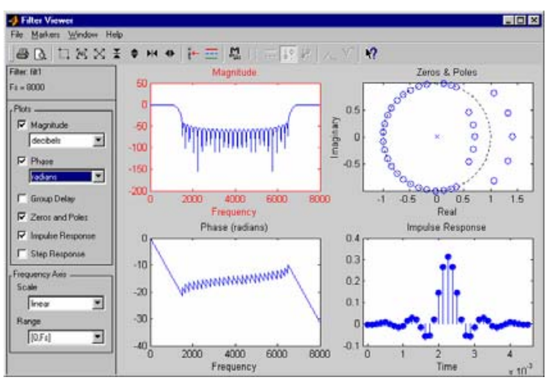
$= 0.4455\,\text{dB}$

Stopband attenuation

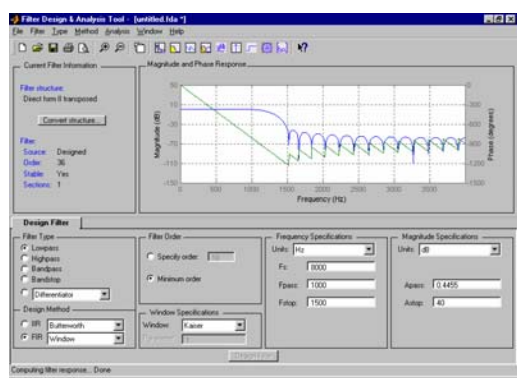
$\Rightarrow R_s = 40\,\text{dB}$



Kaiser Windows Filter Design Method – MATLAB Design Example using sptool



Kaiser Windows Filter Design Method – MATLAB Design Example using fdatool



Kaiser Windows Filter Design Method - MATLAB Design Example

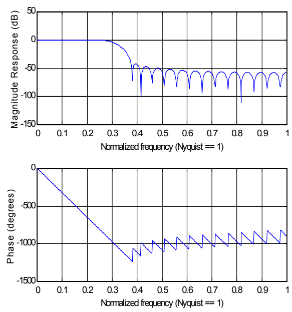
Example 3 Design a lowpass filter with passband from 0 to 1 KHz and stopband from 1500 Hz to 4 KHz. Specify passband ripple of 5% and stopband attenuation of 40 dB.

Alternative Solution 3

Since stopband is ends at 4 kHz, it is implied that sampling freq. is 8 kHz.

Use *kaiserord* and *kaiser* MATLAB functions as follows:

```
» fsamp=8000;
» fcuts=[1000 1500];
» mags=[1 0];
» devs=[0.05 0.01];
» [n,Wn,beta,ftype]=
        kaiserord(fcuts,mags,devs,fsamp);
» hh=fir1(n,Wn,ftype,kaiser(n+1,beta),'noscale');
» freqz(hh)
```



Example: Kaiser Window Design of a Lowpass Filter

- Specifications $\omega_p = 0.4\pi, \omega_p = 0.6\pi, \delta_1 = 0.01, \delta_2 = 0.001$
- Window design methods assume $\delta_1 = \delta_2 = 0.001$
- Determine cut-off frequency
  - Due to the symmetry we can choose it to be $\omega_c = 0.5\pi$
- Compute

$$\Delta\omega = \omega_s - \omega_p = 0.2\pi \qquad A = -20\log_{10}\delta = 60$$

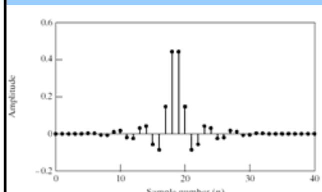
- And Kaiser window parameters

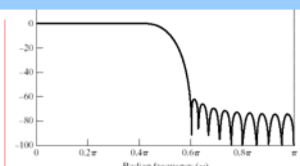
$$\beta = 5.653 \qquad M = 37$$

- Then the impulse response is given as

$$h[n] = \begin{cases} \dfrac{\sin[0.5\pi(n-18.5)]}{\pi(n-18.5)} \dfrac{I_0\left[5.653\sqrt{1-\left(\dfrac{n-18.5}{18.5}\right)^2}\right]}{I_0(5.653)} & 0 \le n \le M \\ 0 & else \end{cases}$$

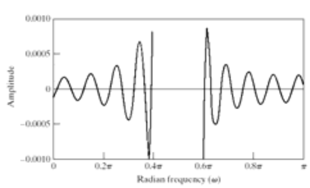
Example Cont'd





Approximation Error

$$E_A(\omega) = \begin{cases} 1 - A_e(e^{j\omega}), & 0 \le \omega \le \omega_p \\ 0 - A_e(e^{j\omega}), & \omega_s \le \omega \le \pi \end{cases}$$

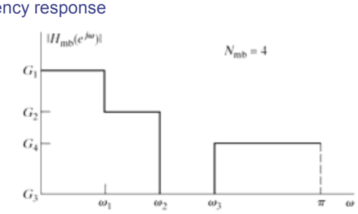


General Frequency Selective Filters

- A general multiband impulse response can be written as

$$h_{mb}[n] = \sum_{k=1}^{N_{mb}} (G_k - G_{k+1}) \frac{\sin\omega_k(n - M/2)}{\pi(n - M/2)}$$

- Window methods can be applied to multiband filters
- Example multiband frequency response
  - Special cases of
    - Bandpass
    - Highpass
    - Bandstop



Linear Phase FIR Filter Design using Windows: Drawbacks

- Although it is simple to design a filter using the window method, there are some drawbacks:-
  - It is necessary to find a closed-form expression for $h_d(n)$.
  - Second, for a frequency selective filter, the transition widths between frequency bands, and the ripples within these bands, will be approximately the same. As a result, the method requires that the filter be designed to the tightest tolerances in all of the bands by selecting the smallest transition width and the smallest ripple.
  - Window design filters are not, in general, optimum in the sense that they do not have the smallest possible ripple for a given filter order and a given set of cutoff frequencies.