

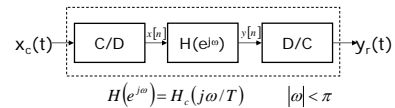
## Filter Design Techniques (IIR Filters)

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## Filter Design Techniques

- Any discrete-time system that modifies certain frequencies
- Frequency-selective filters pass only certain frequencies
- Filter Design Steps
  - Specification
    - Problem or application specific
  - Approximation of specification with a discrete-time system
    - Our focus is to go from spec to discrete-time system
  - Implementation
    - Realization of discrete-time systems depends on target technology
- We already studied the use of discrete-time systems to implement a continuous-time system
  - If our specifications are given in continuous time we can use



## IIR Digital Filter Design Approaches

Two general approaches are used to design IIR digital filters:

✓ *Design an analog IIR filter and then map it into an equivalent discrete filter* The art of analog filter design is highly advanced; thus, it is advantageous to optimally discretise these filters.

✓ *Use an algorithmic design procedure, which generally requires the use of a computer to solve a set of linear or nonlinear equations* These methods may be used to design digital filters with arbitrary frequency response characteristics for which no analog filter prototype exists or to design filters when no other types of constraints are imposed on the design.

## Discrete-Time IIR Filter Design

- Why we want to design discrete-time IIR filters from continuous-time IIR filters
  - The art of continuous-time IIR filter design is highly advanced.
  - Many useful continuous-time IIR filter design methods have relatively simple closed form design formulas.
  - The standard approximation methods that work well for continuous-time IIR filters do not lead to simple closed-form design formulas when these methods are applied directly to the discrete-time IIR case.

### Impulse Invariance for IIR Filter Design

- **Impulse invariance:** a discrete-time system is defined by sampling the impulse response of a continuous-time system by the sampling rate  $T_d$ , i.e.,

$$h[n] = T_d h_c(nT_d)$$

- Relationship between the frequency response of the discrete-time and continuous-time filters

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi k}{T_d}\right)$$

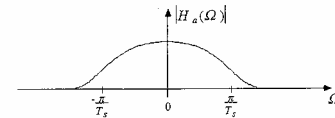
- If the continuous-time filter is band-limited, so that

$$H_c(j\Omega) = 0, |\Omega| \geq \pi/T_d$$

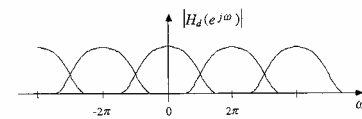
$$\text{Then } H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right), |\omega| \leq \pi \quad \omega = \Omega T_d \text{ for } |\omega| < \pi.$$

### Sampling the Impulse Response (Cont'd)

In practice,  $H_c(j\Omega)$  cannot be strictly bandlimited:



Aliased digital frequency response:



### Impulse Invariance: Aliasing Effect

- It is very difficult to compensate for aliasing effects in the impulse invariance approach. **High sampling rate cannot control aliasing.**
- Usually, we want the continuous-time filters to be **over-designed**, so that aliasing won't affect too much during the sampling process.
- Assume a sufficiently high sampling rate such that  $H_c(j\Omega)$  is effectively band-limited as  $|H_c(j\Omega)| < \epsilon, |\Omega| > \frac{\pi}{T_d}$ .  

$$H_d(e^{j\omega}) \approx H_c\left(j\frac{\omega}{T_d}\right) \approx D(e^{j\omega}); |\omega| \leq \pi.$$
- The scaled analog prototype  $H_c(\omega/T_d)$  must agree satisfactorily with the digital specification  $D(e^{j\omega})$  by selection of, e.g.,  $\delta_p, \delta_s, N$ .
- Aliasing is an important consideration with impulse invariance design. One common strategy is that no more than 10%, or more stringently, 1% of the energy be aliased.

### Impulse Invariance: Implementation

- Let  $H_c(s)$  have  $L$  distinct poles (typical for our prototypes):

$$H_c(s) = \ell\{h_c(t)\} = \sum_{m=1}^L \frac{\alpha_m}{s - p_m}$$

so

$$h_c(t) = \sum_{m=1}^L \alpha_m e^{p_m t} u_a(t); u_a(t) \text{ is the unit step.}$$

- The sampled filter is then

$$h_d[n] = T_s h_c(nT_s) = T_s \sum_{m=1}^L \alpha_m e^{p_m nT_s} u_a(nT_s) = T_s \sum_{m=1}^L \alpha_m e^{p_m nT_s} u_d[n]$$

- The designed digital transfer function is then

$$H_d(z) = T_s \sum_{m=1}^L \frac{\alpha_m}{1 - e^{p_m T_s} z^{-1}}$$

- Which is ideal for a parallel realization of first-order section or even better combined into second-order sections.

### Impulse Invariance: Stability

- If a pole  $p_m$  of the analog filter  $H_a(\Omega)$  lies in the left-hand plane (LHP), then also

$$\operatorname{Re}\{p_m\} < 0 \Rightarrow |e^{p_m T_s}| < 1.$$

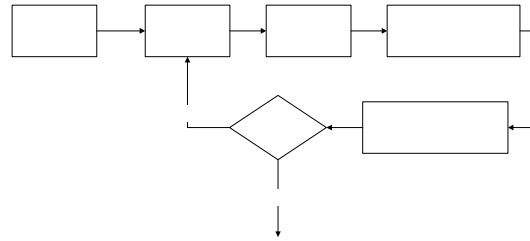
So the corresponding pole of  $H_d(z)$  will lie inside the unit circle. Then the causal filter will be stable.

- Hence the important property

$$\text{Stable } H_a(s) \Leftrightarrow \text{Stable } H_d(z)$$

### Impulse Invariance for IIR Filter Design (Cont'd)

$$\omega = \Omega T_d \text{ for } |\omega| < \pi.$$



### Impulse Invariance for IIR Filter Design: Example

- Let us consider the design of a low-pass discrete-time filter by applying impulse invariance to an appropriate Butterworth continuous-time filter. The specifications for the discrete filter are

$$0.89125 \leq |H_d(e^{j\omega})| \leq 1, \quad 0 \leq |\omega| \leq 0.2\pi$$

$$|H_d(e^{j\omega})| \leq 0.17783, \quad 0.3\pi \leq |\omega| \leq \pi$$

- For simplicity, in the impulse invariance design, we will take  $T_s=1$ , hence we have

$$\omega = \Omega.$$

- Thus, the specifications on the analog filter are

$$0.89125 \leq |H_a(j\Omega)| \leq 1, \quad 0 \leq |\Omega| \leq 0.2\pi$$

$$|H_a(j\Omega)| \leq 0.17783, \quad 0.3\pi \leq |\Omega| \leq \pi$$

- Since the magnitude response of a Butterworth filter is a monotonic function of frequency, the specifications are simplified into

$$0.89125 \leq |H_a(j0.2\pi)| \text{ and } |H_a(j0.3\pi)| \leq 0.17783$$

### Impulse Invariance for IIR Filter Design: Example (Cont'd)

- Specifically, the magnitude function of a Butterworth filter is

$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$$

so we need to determine  $N$  and  $\Omega_c$  to meet the desired specifications which lead to the equations with equality

$$1 + \left(\frac{0.2\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \quad \text{and} \quad 1 + \left(\frac{0.3\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

- The solutions are  $N = 5.8858$  and  $\Omega_c = 0.70474$ . If we use  $N=6$ , then  $\Omega_c = 0.7032$ .

### Impulse Invariance for IIR Filter Design: Example (Cont'd)

- We know that the magnitude function of a Butterworth filter is

$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$$

- Given  $N=6$ , the 12 poles of the magnitude-squared function are uniformly distributed at an angle of  $\pi/6$  on a circle of radius

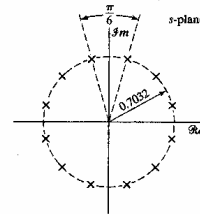
$$\Omega_c = 0.7032$$

$$H_c(s)H_c(-s) = \frac{1}{1 + (s/j\Omega_c)^{2N}}$$

- Consequently, the poles of  $H_c(s)$  are the three pole pairs in the left half of the s-plane.

### Impulse Invariance for IIR Filter Design: Example (Cont'd)

- Pole pair 1:  $-0.182 \pm j(0.679)$ ,
- Pole pair 2:  $-0.497 \pm j(0.497)$ ,
- Pole pair 3:  $-0.679 \pm j(0.182)$ .



s-plane locations for poles of  $H_c(s)H_c(-s)$  for sixth-order

### Impulse Invariance for IIR Filter Design: Example (Cont'd)

- From the 6 poles of  $H_c(s)$ , we have

$$H_c(s) = \frac{0.12093}{(s^2 + 0.3640s + 0.4945)(s^2 + 0.9945s + 0.4945)(s^2 + 1.3585s + 0.4945)}$$

- We use a partial fraction expansion, and we know that

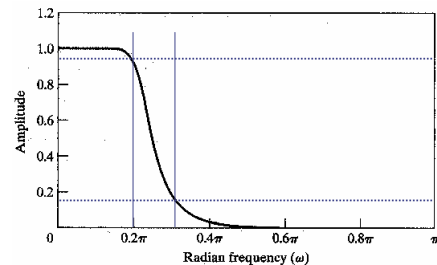
$$H_d(z) = T_s \sum_{m=1}^L \frac{\alpha_m}{1 - e^{p_m T_s} z^{-1}}$$

- We have a parallel form of the digital filter.

$$H_d(z) = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 - 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.8557 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}}$$

### Impulse Invariance for IIR Filter Design: Example (Cont'd)

- The frequency-response functions of the discrete-time filter is shown as follows.



### Bilinear Transformation

- The bilinear transformation is a mapping from the s-plane to the z-plane defined by:
 
$$s = \frac{2}{T_d} \frac{1-z^{-1}}{1+z^{-1}}$$
- Given an analog filter with a transfer function  $H_a(s)$ , the digital filter is designed as follows:
 
$$H(z) = H_a(s) \Big|_{s=\frac{2}{T_d} \frac{1-z^{-1}}{1+z^{-1}}}$$
- The bilinear transformation is a rational function that maps the left-half s-plane inside the unit circle and maps the  $j\Omega$ -axis in a one-to-one manner onto the unit circle.

### Bilinear Transformation $H(z) = H_a(s) \Big|_{s=\frac{2}{T_d} \frac{1-z^{-1}}{1+z^{-1}}}$

- The relationship between analog frequency  $\Omega$  and digital frequency  $\omega$  is highly nonlinear and is given by the *frequency warping function*:
 
$$\omega = 2 \arctan\left(\frac{\Omega T_d}{2}\right)$$
- When  $\Omega=0$ ,  $\omega=0$ , and as  $\Omega \rightarrow \infty$ ,  $\omega \rightarrow \pi$ . The one-to-one mapping nonlinearly compresses the analog frequency range  $-\infty < f < \infty$  to the digital frequency  $-\pi < \Omega < \pi$ .
- It avoids the effects of aliasing at the expense of
  - distorting,
  - compressing
  - warping the analog frequencies - preserve magnitude response
- Therefore, the transformation is generally only used in the design of frequency selective filters.

### Bilinear Transformation Design Steps

The steps involved in the design of a digital lowpass with a passband cutoff frequency  $\omega_p$ , stopband cutoff frequency  $\omega_s$ , passband ripple  $\delta_p$ , and a stopband ripple  $\delta_s$  are as follows:

- Prewarp** (to compensate for the warping) the passband and stopband cutoff frequency specifications of the digital filter,  $\omega_p$  and  $\omega_s$  using the inverse of the *frequency warping function* to determine the passband and stopband cutoff frequencies ( $\Omega_p$  and  $\Omega_s$ ) of the analog lowpass filter. With  $T_s=2$ , the prewarping function is:  $\Omega = \tan\left(\frac{\omega}{2}\right)$
- Design an analog lowpass** filter  $H_a(s)$  with the cutoff (prewarp) frequencies  $\Omega_p$  and  $\Omega_s$  found in step 1 and, Passband and stopband ripples  $\delta_p$  and  $\delta_s$ , respectively.
- Apply the bilinear transformation** to the analog prototype designed in step 2 to obtain the digital filter:  $H(z) = H_a(s) \Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$

### Bilinear Transformation Design Example

**Design Example 1** Design a first-order digital lowpass filter with a 3 dB cutoff frequency of  $\omega_c = 0.25\pi$  by applying the bilinear transformation to the analog Butterworth filter,

$$H_a(s) = \frac{1}{1+s/\Omega_c}$$

**Solution**

Since the 3-dB cutoff frequency of the Butterworth filter is  $\Omega_c$ , for a cutoff frequency  $\omega_c = 0.25\pi$  in the digital filter, we must have

$$\Omega_c = \frac{2}{T_d} \tan\left(\frac{0.25\pi}{2}\right) = \frac{0.828}{T_d}$$

Therefore, the transfer function of the analog filter is  $H_a(s) = \frac{1}{1+sT_d/0.828}$

Applying the bilinear transformation to the analog filter gives

$$H(z) = H_a(s) \Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{1}{1+(2/0.828)[(1-z^{-1})/(1+z^{-1})]} = 0.2920 \frac{1+z^{-1}}{1-0.4159z^{-1}}$$

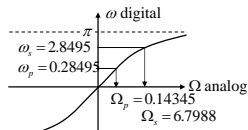
Note the parameter  $T_d$  does not enter into the design.

## Bilinear Transformation Design Example

**Design Example 2** Design a digital Butterworth lowpass filter operating at 44.1 kHz with a 1-dB cutoff frequency at 2 kHz. The required minimum stopband attenuation is 50 dB at 20 kHz.

### Solution

1. Prewarp the digital filter frequency specifications to obtain the corresponding analog prototype filter specifications:



$$\omega_p = \frac{2k}{44.1k} \cdot 2\pi = 0.090702947\pi = 0.284951714$$

$$\Rightarrow \Omega_p = \frac{2}{T_s} \tan\left(\frac{0.284951714}{2}\right)$$

Let  $T_s = 2$

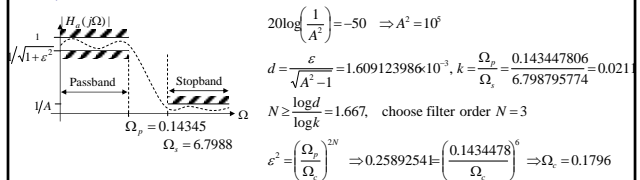
$$\Rightarrow \Omega_p = \tan\left(\frac{0.284951714}{2}\right) = 0.143447806$$

$$\omega_s = \frac{20k}{44.1k} \cdot 2\pi = 0.90702947\pi = 2.849517146$$

$$\Rightarrow \Omega_s = \tan\left(\frac{2.849517146}{2}\right) = 6.798795774$$

## Bilinear Transformation Design Example

2. Design analog prototype using the  $\Omega_p$  and  $\Omega_s$ :



Based on Butterworth filter coefficients table, the normalised 3<sup>rd</sup> order Butterworth analog filter:

$$H_3(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \Rightarrow H_3(s/\Omega_c) \Big|_{\Omega_c=0.1796} = \frac{1}{\left(\frac{s}{0.1796}\right)^3 + 2\left(\frac{s}{0.1796}\right)^2 + 2\left(\frac{s}{0.1796}\right) + 1}$$

3. The digital filter is:

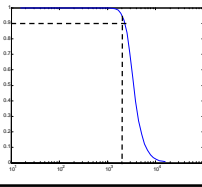
$$H(z) = H_3(s/0.1796) \Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

## Bilinear Transformation MATLAB Design Example

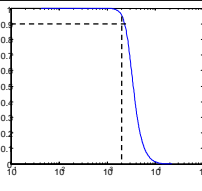
```
% analog design
Wp = 2*pi*2000; Ws = 2*pi*20000;
[n,Wn] = buttord(Wp,Ws,1,50,'s'); % get filter order & centre
freq
[b,a] = butter(n,Wn,'s'); % Butterworth filter coeffs
w = logspace(2,5); % frequencies 10 to 100K
rad/sec
h = freqz(b,a,w); %
mag = abs(h); f = w/(2*pi);
subplot(1,1,1), semilogx(f, mag)
```

$$H_s(s) = \frac{B(s)}{A(s)} = \frac{b(1)s^2 + b(2)s^1 + \dots + b(n+1)}{s^2 + a(2)s^1 + \dots + a(n+1)}$$

$$H(z) = \frac{B(z)}{A(z)} = \frac{b(1)+b(2)z^{-1}+\dots+b(n+1)z^{-n}}{1+a(2)z^{-1}+\dots+a(n+1)z^{-n}}$$



```
% digital design - bilinear transformation used in MATLAB
Fsamp = 44100; Fnyquist = Fsamp/2;
Fp = 2000/Fnyquist; Fs = 20000/Fnyquist;
[n,Wn] = buttord(Fp,Fs,1,50,'s'); % filter order & centre freq
[b,a] = butter(n,Wn); % prewarp & bilinear tx
internally
[h f] = freqz(b,a,512,Fsamp);
mag = abs(h);
subplot(1,1,1), semilogx(f, mag)
```



## Example

• Bilinear transform applied to Butterworth

$$0.89125 \leq |H(e^{j\omega})| \leq 1 \quad 0 \leq |\omega| \leq 0.2\pi$$

$$|H(e^{j\omega})| \leq 0.17783 \quad 0.3\pi \leq |\omega| \leq \pi$$

• Apply bilinear transformation to specifications

$$0.89125 \leq |H(j\Omega)| \leq 1 \quad 0 \leq |\Omega| \leq \frac{2}{T_d} \tan\left(\frac{0.2\pi}{2}\right)$$

$$|H(j\Omega)| \leq 0.17783 \quad \frac{2}{T_d} \tan\left(\frac{0.3\pi}{2}\right) \leq |\Omega| < \infty$$

• We can assume  $T_d=1$  and apply the specifications to

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

• To get

$$1 + \left(\frac{2 \tan 0.1\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \quad \text{and} \quad 1 + \left(\frac{2 \tan 0.15\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

### Example Cont'd

- Solve  $N$  and  $\Omega_c$

$$N = -\frac{\log\left[\left(\left(\frac{1}{0.17783}\right)^2 - 1\right) / \left(\left(\frac{1}{0.89125}\right)^2 - 1\right)\right]}{2 \log[\tan(0.15\pi) / \tan(0.1\pi)]} = 5.305 \cong 6 \quad \Omega_c = 0.766$$

- The resulting transfer function has the following poles

$$s_k = (-1)^{k/12} (j\Omega_c) = \Omega_c e^{(j\pi/12)(2k+1)} \text{ for } k = 0, 1, \dots, 11$$

- Resulting in

$$H_c(s) = \frac{0.20238}{(s^2 + 0.3996s + 0.5871)(s^2 + 1.0836s + 0.5871)(s^2 + 1.4802s + 0.5871)}$$

- Applying the bilinear transform yields

$$H(z) = \frac{0.0007378(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})} \times \frac{1}{(1-0.9044z^{-1}+0.2155z^{-2})}$$

### Example Cont'd

